1(i) $(|\vec{v}|\vec{u}+|\vec{u}|\vec{v})\cdot(|\vec{v}|\vec{u}-|\vec{u}|\vec{v})$

$= (|\vec{v}|\vec{u}+|\vec{u}|\vec{v})\cdot|\vec{v}|\vec{u} - (|\vec{v}|\vec{u}+|\vec{u}|\vec{v})\cdot|\vec{u}|\vec{v}$

$= |\vec{v}|^2|\vec{u}|^2 + |\vec{u}||\vec{v}|\,\vec{v}\cdot\vec{u} - |\vec{v}||\vec{u}|\,\vec{u}\cdot\vec{v} - |\vec{u}|^2|\vec{v}|^2$

$= 0$

$\therefore$ $|\vec{v}|\vec{u}+|\vec{u}|\vec{v}$ and $|\vec{v}|\vec{u}-|\vec{u}|\vec{v}$

are perpendicular to each other.

1(ii) $$\vec{u}\cdot\vec{w} = \frac{|\vec{v}|}{|\vec{u}|+|\vec{v}|}|\vec{u}|^2 + \frac{|\vec{u}|}{|\vec{u}|+|\vec{v}|}\vec{v}\cdot\vec{u}$$

$$\frac{\vec{u}\cdot\vec{w}}{|\vec{u}|} = \frac{|\vec{u}||\vec{v}|}{|\vec{u}|+|\vec{v}|} + \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|+|\vec{v}|}$$

Similarly, $$\frac{\vec{v}\cdot\vec{w}}{|\vec{v}|} = \frac{|\vec{u}||\vec{v}|}{|\vec{u}|+|\vec{v}|} + \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|+|\vec{v}|}$$

$$\therefore \frac{\vec{u}\cdot\vec{w}}{|\vec{u}||\vec{w}|} = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\cdot|\vec{w}|} \quad \text{—— ①}$$

Let θ_1, θ_2 be the included angles of vectors $\vec{u},\vec{w}$ and of vectors $\vec{v},\vec{w}$ respectively. $(0 \le \theta_1, \theta_2 \le \pi)$

Then, ① becomes

$$\cos\theta_1 = \cos\theta_2$$

$$\theta_1 = \theta_2 \quad \#$$

2(i) $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$

$|\vec{u}-\vec{v}|^2 = |\vec{u}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$

$\therefore \quad |\vec{u}+\vec{v}|^2 - |\vec{u}-\vec{v}|^2 = 4\vec{u}\cdot\vec{v}$

$$\vec{u}\cdot\vec{v} = \frac{1}{4}\left(|\vec{u}+\vec{v}|^2 - |\vec{u}-\vec{v}|^2\right)$$

2 (ii)

$$|\vec{u}\times\vec{v}|^2 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2$$

$$= (u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2$$

$$= u_2^2v_3^2 + u_3^2v_2^2 - 2u_2v_2u_3v_3 + u_1^2v_3^2 + u_3^2v_1^2 - 2u_1v_1u_3v_3 + u_1^2v_2^2 + u_2^2v_1^2 - 2u_1v_1u_2v_2$$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$$

$$- u_1^2 v_1^2 - u_2^2 v_2^2 - u_3^2 v_3^2$$

$$- 2u_2 v_2 u_3 v_3 - 2u_1 v_1 u_3 v_3 - 2u_1 v_1 u_2 v_2$$

$$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2)$$

$$- (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

$$= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$

3(i) In $\mathbb{R}^3$, by 2(ii)

$$|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2$$

and $|\vec{u} \times \vec{v}|^2 \geq 0$.

$$\therefore \quad |\vec{u}|^2 \cdot |\vec{v}|^2 \geq (\vec{u} \cdot \vec{v})^2$$

$$\therefore \quad |\vec{u} \cdot \vec{v}| \leq |\vec{u}| \cdot |\vec{v}|.$$

In $\mathbb{R}^2$, if $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$,

then we may consider $(u_1, u_2, 0)$, $(v_1, v_2, 0)$

in $\mathbb{R}^3$. Above gives us that

$$|u_1 v_1 + u_2 v_2| \leq \sqrt{u_1^2 + u_2^2} \sqrt{v_1^2 + v_2^2}$$

3 (ii) $|\vec{u}+\vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u}\cdot\vec{v}$

and $(|\vec{u}|+|\vec{v}|)^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2|\vec{u}||\vec{v}|$

By 3(i), we have

$$|\vec{u}|\cdot|\vec{v}| \geq |\vec{u}\cdot\vec{v}|$$

$\therefore$ $|\vec{u}+\vec{v}|^2 \leq (|\vec{u}| + |\vec{v}|)^2$

i.e. $|\vec{u}+\vec{v}| \leq |\vec{u}| + |\vec{v}|$

3 (iii) If we put

$$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n$$

and $|\vec{u}| = |\vec{u}\cdot\vec{u}|^{\frac{1}{2}}$ for any $\vec{u}, \vec{v}$

then for both $\vec{u}, \vec{v} \neq 0$.

$$\left|\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}\right|^2 = \left(\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}\right)\cdot\left(\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|}\right)$$

$$= \frac{\vec{u}}{|\vec{u}|}\cdot\frac{\vec{u}}{|\vec{u}|} - \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} - \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

$$+ \frac{\vec{v}}{|\vec{v}|}\cdot\frac{\vec{v}}{|\vec{v}|}$$

$$= 2 - 2\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$$

Since $\left| \frac{\vec{u}}{|\vec{u}|} - \frac{\vec{v}}{|\vec{v}|} \right|^2 \geq 0$,

we have

$$\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \leq 1$$

i.e. $\vec{u} \cdot \vec{v} \leq |\vec{u}||\vec{v}|$ —— ②

② also holds if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

Since ② is true for any $\vec{u}, \vec{v}$.

if we replace $\vec{u}$ by $-\vec{u}$, we have

$$-\vec{u} \cdot \vec{v} \leq |\vec{u}| \cdot |\vec{v}|$$

$\therefore$ $\vec{u} \cdot \vec{v} \geq -|\vec{u}||\vec{v}|$

$\therefore$ $|\vec{u} \cdot \vec{v}| \leq |\vec{u}||\vec{v}|$ for any $\vec{u}, \vec{v}$.

4. Take P as a reference point,

$\vec{PQ} = \vec{OQ} - \vec{OP} = (1,0,4) - (1,1,0)$

$= (0,-1,4)$

$\vec{PR} = \vec{OR} - \vec{OP} = (0,2,5) - (1,1,0)$

$= (-1,1,5)$

$$\text{Area of } \Delta PQR = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}|$$

Note that

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 4 \\ -1 & 1 & 5 \end{vmatrix}$$

$$= \langle -9, -4, -1 \rangle$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{9^2 + 4^2 + 1^2}$$

$$= \sqrt{98} = 7\sqrt{2}$$

$$\therefore \text{ Area of } \Delta PQR = \frac{7\sqrt{2}}{2}$$

5. Take P as a reference point.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (4, 2, -2) - (2, 0, 0)$$

$$= (2, 2, -2)$$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = (-7, 0, 4)$$

$$\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = (2, -5, 1)$$

Notice that

$$\begin{aligned}\overrightarrow{PS} &= (2, -5, 1) \\ &= -\frac{5}{2}(2, 2, -2) - (-7, 0, 4) \\ &= -\frac{5}{2}\overrightarrow{PQ} - \overrightarrow{PR}\end{aligned}$$

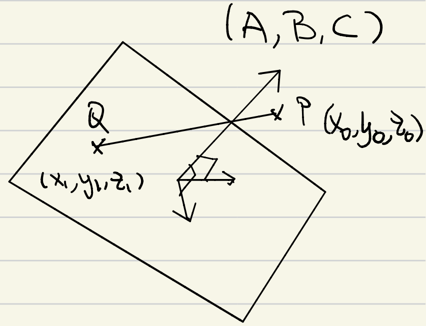
$\therefore$ S is in the plane formed by P, Q, R

i.e. P, Q, R, S are coplaner.

This one can be replaced by showing that $\overrightarrow{PS} \cdot (\overrightarrow{PQ} \times \overrightarrow{PR}) = 0$

6. Let $Q(x_1, y_1, z_1)$ be a point on the plane

(A, B, C) is a normal vector of the plane



The distance of P is the plane is the length of projection of $\overrightarrow{PQ}$ on the normal vector (A,B,C)

That is, $\left| \dfrac{\overrightarrow{PQ}\cdot(A,B,C)}{|(A,B,C)|^2}(A,B,C) \right|$

$$= \frac{|\overrightarrow{PQ}\cdot(A,B,C)|}{|(A,B,C)|}$$

$$= \frac{|A(x_1-x_0)+B(y_1-y_0)+C(y_1-y_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax_1+By_1+Cz_1-(Ax_0+By_0+Cy_0)|}{\sqrt{A^2+B^2+C^2}}$$

$$= \frac{|Ax_0+By_0+Cy_0+D|}{\sqrt{A^2+B^2+C^2}}$$

$\because$ (x_1, y_1, z_1) satisfies the equation of the plane.

7. For cross product, the direction is determined by right hand rule, and the magnitude is $|\vec{u}| \cdot |\vec{v}| \, |\sin\theta|$.

We may let $\vec{u} = (1,0,0)$, $\vec{v} = (0,1,0)$

and $\vec{w} = 2(\cos\theta, \sin\theta, 0)$

$= 2\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$ if $\theta = \frac{\pi}{6}$

$= (\sqrt{3}, 1, 0)$

$\vec{u} \times \vec{v} = \hat{k}$

$$\vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ \sqrt{3} & 1 & 0 \end{vmatrix} = \hat{k}$$

8. For the line ℓ:

$$\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-2}{4}$$

It passes through $(-1, 1, 2)$ with direction $(3, 2, 4)$

Normal vector of the plane $2x+y-3z+4=0$

is $(2,1,-3)$

$\therefore$ The required plane is parallel to

both $(3,2,4)$ and $(2,1,-3)$. Moreover,

it passes through the point $(-1,1,2)$.

Normal vector of the required plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 4 \\ 2 & 1 & -3 \end{vmatrix} = \langle -10, 17, -1 \rangle$$

So, the equation of plane is

$$-10x + 17y - z = D$$

for some $D \in \mathbb{R}$.

Plugging in $(-1,1,2)$, we have $D = 25$

$\therefore$ Equation of the plane is

$$10x - 17y + z + 25 = 0$$

9. Let $\vec{v} = \vec{c} \times \vec{d} = (v_1, v_2, v_3)$

We may show that $(\vec{a} \times \vec{b}) \times \vec{v}$ is a linear combination of $\{\vec{a}, \vec{b}\}$

For $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1)$$

Hence, $(\vec{a} \times \vec{b}) \times \vec{v}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_2b_3 - a_3b_2 & -(a_1b_3 - a_3b_1) & a_1b_2 - a_2b_1 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \Big(-v_3(a_1b_3 - a_3b_1) - v_2(a_1b_2 - a_2b_1),$$

$$- v_3(a_2b_3 - a_3b_2) + v_1(a_1b_2 - a_2b_1),$$

$$v_2(a_2b_3 - a_3b_2) + v_1(a_1b_3 - a_3b_1) \Big)$$

$$= \big(-a_1(b_2v_2 + b_3v_3) + b_1(a_2v_2 + a_3v_3),$$
$$-a_2(b_1v_1 + b_3v_3) + b_2(a_1v_1 + a_3v_3),$$
$$-a_3(b_1v_1 + b_2v_2) + b_3(a_1v_1 + a_2v_2) \big)$$

$$= \big(-a_1(b_1v_1 + b_2v_2 + b_3v_3) + b_1(a_1v_1 + a_2v_2 + a_3v_3),$$
$$-a_2(b_1v_1 + b_2v_2 + b_3v_3) + b_2(a_1v_1 + a_2v_2 + a_3v_3),$$
$$-a_3(b_1v_1 + b_2v_2 + b_3v_3) + b_3(a_1v_1 + a_2v_2 + a_3v_3) \big)$$

$$= -(\vec{b} \cdot \vec{v})\vec{a} + (\vec{a} \cdot \vec{v})\vec{b}$$

$\therefore$ $(\vec{a} \times \vec{b}) \times \vec{v}$ is a linear combination of $\{\vec{a}, \vec{b}\}$

This result can imply that $(\vec{c} \times \vec{d}) \times (\vec{a} \times \vec{b})$ is a linear combination of $\{\vec{c}, \vec{d}\}$. Therefore,

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = -(\vec{c} \times \vec{d}) \times (\vec{a} \times \vec{b})$

is also a linear combination of $\{\vec{c}, \vec{d}\}$

10. Suppose L_1, L_2 are parallel.

Then, L_1, L_2 are not skew,

Moreover, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

because (a_1, b_1, c_1) is a scaler multiple of (a_2, b_2, c_2) in this case.

Therefore, we may assume that L_1, L_2 are not parallel from now on.

Note that in Tutorial notes (week 2), we have seen that

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

is the equation of plane, which contains L_1 and is parallel to L_2

$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

iff the plane also contains L_2

That is

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

(A) iff there is a plane containing both L_1 and L_2

Suppose two lines have some intersection, say (x_0, y_0, z_0), Then,

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

is the plane containing both L_1 and L_2

By (A),

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Suppose two lines have no intersection,

i.e. for any $t, s \in \mathbb{R}$,

$$(x_1, y_1, z_1) + t(a_1, b_1, c_1) \neq (x_2, y_2, z_2) + s(a_2, b_2, c_2)$$

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1) \neq t(a_1, b_1, c_1) - s(a_2, b_2, c_2)$$

$\therefore$ $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is not a linear combination of $\{(a_1, b_1, c_1), (a_2, b_2, c_2)\}$

$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \neq 0$$